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Order-Independent Method of Characteristics

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A method of characteristics is developed for any system of partial differential equations of any finite order that admits an isovector field V and an initial data map satisfying a specific transversality condition. It is shown to agree with the classical method of characteristics for a nonlinear, first-order PDE and for quasilinear systems of first-order PDE with the same principal part. The method is also applicable to systems of nonlinear, first-order PDE and to systems of higher order, where it agrees with results obtained by similarity and group invariant methods. Implementation of the characteristic method is easier than classical group invariant methods because a complete, independent system of invariants of the flow generated by the isovector (group symmetry) does not have to be computed. General solutions are obtained only when V is a Cauchy characteristic vector of the fundamental ideal; otherwise, any characteristic solution is shown to satisfy an explicit system of differential constraints. Explicit examples and comparisons with more classical methods are given.

1. INTRODUCTION AND SUMMARY

Symmetry methods for partial differential equations (PDE) have been around since the time of S. Lie, while the classical method of characteristics for a first-order PDE has been in use a good deal longer. Recent work from the standpoint of isovector fields of differential ideals has shown that these two methods are actually two facets of the same geometric construct, and that a general method of characteristics exists for second-order and systems of nonlinear first-order PDE. The purposes of this paper are to give explicit derivations, examples, and proofs of the results that underlie this order-independent method of characteristics.

Let M_n be the n-dimensional manifold of independent variables, and let $\{x^i | 1 \leq i \leq n\}$ be a local coordinate cover. In practice, n will be 4 or less, and I will usually write $\{x, y, z, t\}$ or a subset of these in order to simplify

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the examples. The volume element (basis n-form) of M_n will be denoted by μ. For $n=4$, we have $\mu = dx \wedge dy \wedge dz \wedge dt$. The conjugate basis for $(n-1)$-forms is given by $\{\mu_i = \partial_i \rfloor \mu | 1 \leq i \leq n\}$ with the properties (Edelen, 1985, Section 3.5)

$$dx^j \wedge \mu_i = \delta^j_i \mu, \qquad d\mu_i = 0$$

The N-dimensional range space for the dependent variables will be denoted by E_N, and I will use a local system of coordinates $\{q^\alpha | 1 \leq \alpha \leq N\}$. The $(n+N)$-dimensional space $G = M_n \times E_N$ will be referred to as *graph space* for obvious reasons. It is where we plot the graphs of the dependent variables as functions of the independent variables for a given evaluation $q^\alpha = \phi^\alpha(x^i)$. Such an evaluation is most easily envisioned as a mapping Φ from M_n to G of the form

$$\Phi: \quad M_n \to G \quad | \quad x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^i) \tag{1.1}$$

The study of PDE requires place holders for the various partial derivatives of the dependent variables. These are provided by introducing a *contact manifold* $K = G \times \mathbb{R}^m$ with local coordinates $\{x^i, q^\alpha, y^\alpha_i, y^\alpha_{ij}, \ldots\}$ and *contact 1-forms*

$$C^\alpha = dq^\alpha - y^\alpha_k \, dx^k \tag{1.2}$$

$$C^\alpha_i = dy^\alpha_i - y^\alpha_{ik} \, dx^k \tag{1.3}$$

$$C^\alpha_{ij} = dy^\alpha_{ij} - y^\alpha_{ijk} \, dx^k, \ldots \tag{1.4}$$

If a map $\Phi: M_n \to K$ annihilates each of the contact 1-forms, then the y's become the derivatives of the dependent variables with respect to the independent variables (Edelen, 1985, Chapter 6). The PDE under study can then be written as a collection of elements $\{H^a | 1 \leq a \leq r\}$ of $\Lambda(K)$. For example, with $n=2$, $N=1$, the PDE $\phi\, \partial_x \phi - \partial_t \partial_t \phi = 0$ could be written as $H^1 = q y_x - y_{tt}$ if we use second-order contact forms, or by $H^1 = q\, dq \wedge dt + dy_t \wedge dx$ if only first-order contact forms are used.

This information can be organized in a more efficient manner by using the fact that $\Lambda(K)$ is a graded algebra and thus has well-defined ideals. The *fundamental ideal* $\mathcal{I}$ of a given system of PDE is the closed differential ideal generated by the contact 1-forms and the H's,

$$\mathcal{I} = I\{C^\alpha, dC^\alpha, C^\alpha_i, dC^\alpha_i, \ldots, H^a, dH^a\} \tag{1.5}$$

The collection of all *solution maps* of the given system of PDE is given by

$$S = \{\Phi: M_n \to K \quad | \quad \Phi^* \mu \neq 0, \Phi^* \mathcal{I} = 0\} \tag{1.6}$$

The requirement $\Phi^*\mu \neq 0$ guarantees that the range of Φ in K projects onto the base manifold M_n as an n-dimensional region; that is, the x's remain independent of the range of Φ. On the other hand, $\Phi^*\mathscr{I} = 0$ if and only if

$$\Phi^* C^\alpha = 0, \qquad \Phi^* C^\alpha_i = 0, \ldots, \tag{1.7}$$

and

$$\Phi^* H^a = 0 \tag{1.8}$$

because $\Phi^*\Omega = 0$ implies $\Phi^* d\Omega = 0$.

The discussion of vector fields on K can be simplified if one introduces a system of collective coordinates by

$$\{z^A\} = \{x^i, q^\alpha, y^\alpha_i, y^\alpha_{ij}, \ldots\} \tag{1.9}$$

where the index A ranges over the integers from 1 through $m = \dim(K)$. A vector field on K has a representation as a derivation

$$V = v^A(z^B)\partial_A, \qquad \partial_A := \frac{\partial}{\partial z^A} \tag{1.10}$$

It may also be thought of as the field of tangents to the orbits of V; namely, tangents to the curves in K that are defined by solving the *orbital equations*

$$\frac{dZ^A(s)}{ds} = v^A(Z^B(s)), \qquad Z^A(0) = z^A \tag{1.11}$$

The solutions of (1.11) define a one-parameter family of automorphisms of K by

$$T_V(s): \quad K \to K \quad | \quad {}'z^A = Z^A(s) \tag{1.12}$$

These automorphisms move the points of K along the orbits of V (along the flow generated by V). Thus, if Φ is a map from M_n into K, we can compose $T_V(s)$ with Φ to obtain a one-parameter family of maps

$$\Phi_V(s) = T_V(s) \circ \Phi \tag{1.13}$$

Simply use the range of Φ as initial data for the orbital equations of V. This family has the obvious property that

$$\Phi_V(0) = \Phi \tag{1.14}$$

and hence (1.13) defines an embedding of the given map Φ in a one-parameter family of maps.

If Φ is a solution map of the given system of PDE (i.e., $\Phi \in S$), we would obviously like to choose the vector field V so that $\Phi_V(s)$ is also a solution map for each value of s in some neighborhood $\mathbb{J}$ of $s = 0$, for we would then have embedded our known solution in a one-parameter family

of solutions. Vector fields with this property are referred to as *isovectors* of the fundamental ideal $\mathcal{I}$. They were first introduced in the seminal paper by Harrison and Estabrook (1971). The collection of all isovectors of the fundamental ideal $\mathcal{I}$ is denoted by TI. The fundamental theorem for isovectors (Edelen, 1980, 1985) shows that TI forms a Lie subalgebra of $T(K)$ over $\mathbb{R}$ and that

$$TI = \{V \in T(K) \quad | \quad \mathcal{L}_V \mathcal{I} \subset \mathcal{I}\} \tag{1.15}$$

where $\mathcal{L}$ denotes the Lie derivative. Isovectors for numerous important systems of PDE are reported in the literature (Edelen, 1980, 1983, 1985; Harrison and Estabrook, 1971; Papachristov and Harrison, in press; Chowdhury, 1986; Delph, 1983). Isovectors can also be obtained from knowledge of the infinitesimal generators of symmetry groups of systems of PDE that are computed by classical group extension methods (Blumin and Cole, 1974; Ovsiannikov, 1982; Ames, 1982; Ibragimov, 1985), or by prolongation methods in jet bundle formulations (Olver, 1986; Pommaret, 1978). I will therefore assume that at least a one-dimensional vector subspace of isovectors of the fundamental ideal is known.

Knowledge of a $V \in TI$ does not provide new solutions to a given system of PDE unless we have at least one solution map Φ with which to start the process (you don't get something for nothing). The reader should also note that if $\Phi = \Phi_V(0)$ is a solution map and $V \in TI$, then $\Phi_V(s)$ is a solution map only for each fixed numerical value of s in $\mathbb{J}$. Thus, changing the value of s in (1.13) takes us from one solution of the given system of PDE to a neighboring solution, but s cannot be allowed to change in the process of obtaining a solution $\Phi_V(s)$ from a known solution Φ.

What happens when s is allowed to vary is reported in the remainder of this paper. It leads to a generalized method of characteristics that is applicable to any system of PDE that can be characterized in terms of a fundamental ideal $\mathcal{I}$ of $\Lambda(K)$. This generalized method of characteristics agrees with the classical method for a single, nonlinear, first-order PDE and for quasilinear systems of first-order PDE with the same principal part (see Sections 5 and 6). Here, we obtain general solutions (i.e., there are no additional constraints imposed by the method). The generalized method of characteristics also applies to systems of nonlinear, first-order PDE and to systems of second- and higher-order PDE. Explicit examples are given in Sections 5 and 6. In fact, this method of characteristics is an order-independent method that depends only on knowledge of an isovector of the fundamental ideal that satisfies a specific transversality condition. There is a price to be paid in the case of a general fundamental ideal, for we only obtain solutions that satisfy certain explicit constraints that are determined by the isovector field that is used. These results agree with those that can be

obtained by looking for group invariant (similarity) solutions, but are much easier to implement, since we do not have to obtain a complete system of scalar invariants of the orbital equations. These matters are discussed in Section 8.

The specifics of the order-independent method of characteristics are as follows. The reader is referred to Sections 2, 3, and 6 for derivations and proofs. I will use generic coordinates $\{z^A\} = \{x^i, q^\alpha, y^\alpha_i, \ldots\}$ on K for this discussion. Let

$$V = v^A(z^B)\partial_A \tag{1.16}$$

be an isovector of the fundamental ideal $\mathscr{I}$. I introduce an $(n-1)$-dimensional parameter space $\mathbb{R}^{n-1}$ and use D_{n-1} to denote a connected, open subset of $\mathbb{R}^{n-1}$ with local coordinates $\{u^l \mid 1 \le l \le n-1\}$. A map

$$\psi: \quad D_{n-1} \to K \quad | \quad z^A = \psi^A(u^l) \tag{1.17}$$

may be thought of as defining an *initial data manifold* in K that is parametrized by the parameters $\{u^l\}$. A system of functions $\{Z^A(u^l; \tau)\}$ can be defined by solving the initial value problem

$$\frac{dZ^A}{d\tau} = v^A(Z^B), \qquad Z^A(0) = \psi^A(u^l) \tag{1.18}$$

The map

$$\Psi: \quad D_{n-1} \times \mathbb{R} \to K \quad | \quad z^A = Z^A(u^l; \tau) \tag{1.19}$$

is a solving map of the fundamental ideal $\mathscr{I}$ if the pair (V, ψ) satisfies the *transversality condition*

$$\psi^*(V \rfloor \mu) \neq 0 \tag{1.20}$$

and the *initial data conditions*

$$\psi^* \mathscr{I} = 0, \qquad \psi^*(V \rfloor \mathscr{I}) = 0 \tag{1.21}$$

where $\rfloor$ denotes the standard inner product. Here, by (1.21) I mean that ψ^* annihilates each of the generators of $\mathscr{I}$ and also annihilates $V\rfloor$ acting on each of the generators of $\mathscr{I}$.

These conditions may seem unduly complicated on first reading. The geometric picture is rather simple, however. If (V, ψ) satisfies the transversality condition (1.20), then the orbits of the vector field V are not tangent to the initial data manifold defined by ψ, and ψ has rank $n-1$. If the initial data defined by ψ are then chosen so as to satisfy the initial data conditions (1.21), then we simply pull these initial data long the orbits of V with orbital parameter τ in order to generate the n-dimensional region of K that is the characteristic solution defined by the map Ψ.

There are also four rather drastic simplifications that obtain. First, isovectors of the fundamental ideal $\mathscr{I}$ are usually determined by their projection onto graph space [i.e., they are prolongations (Edelen, 1985; Olver, 1986) of vector fields on graph space],

$$V_G = v^i(x^j, q^\beta)\partial_i + v^\alpha(x^j, q^\beta)\partial_\alpha \tag{1.22}$$

In this case, we only have to solve the reduced orbital equations

$$\frac{dX^i}{d\tau} = v^i(X^j, Q^\beta), \qquad X^i(0) = \psi^i(u^l) \tag{1.23}$$

$$\frac{dQ^\alpha}{d\tau} = v^\alpha(X^j, Q^\beta), \qquad Q^\alpha(0) = \psi^\alpha(u^l) \tag{1.24}$$

and the characteristic solution is then give in implicit parametric form by

$$x^i = X^i(u^l; \tau), \qquad q^\alpha = Q^\alpha(u^l; \tau) \tag{1.25}$$

Second, the components of V_G are often affine functions of their $n+N$ arguments, in which case the reduced orbital equations (1.23) and (1.24) can be solved in closed form! Whenever closed-form solutions of the orbital equations can be obtained, (1.25) will provide exact solutions of the system of PDE. Otherwise, the orbital equations will have to be integrated numerically, in which case the functions $X^i(u^l; \tau)$ and $Q^\alpha(u^l; \tau)$ will only be approximate. Numerical integration is permitted in the generalized method of characteristics because we are able to compute the initial data constraints (reduced field equations) without having to integrate the orbital equations [see (1.21)].

Third, the initial data conditions (1.21) can be implemented whether or not the orbital equations can be explicitly solved! This is in strong contrast to similarity methods and the method of group invariant solutions (Olver, 1986, Chapter 3), since these methods require a complete system of primitive integrals of the orbital equations. The reader is referred to Section 8 for a discussion of the implications of this fundamental difference between the method of characteristics and the standard method of obtaining group invariant (similarity) solutions.

Fourth, the initial data conditions (1.21) always simplify. In order to see this, let us note that the fundamental ideal $\mathscr{I}$ is generated by a given collection $\{\Omega_k \,|\, 1 \leq k \leq M\}$ of differential forms,

$$\mathscr{I} = I\{\Omega_k, d\Omega_k \,|\, 1 \leq k \leq M\}$$

The initial data conditions (1.21) are satisfied if and only if

$$\psi^*\Omega_k = 0, \qquad \psi^*(V \rfloor \Omega_k) = 0 \tag{1.26}$$

$$\psi^* d\Omega_k = 0, \qquad \psi^*(V \rfloor d\Omega_k) = 0 \tag{1.27}$$

Since d and ψ^* commute, the first of (1.26) implies the first of (1.27) for each value of the index m. Further, since V is an isovector of $\mathscr{I}$, $\mathscr{L}_V\Omega_k = V\rfloor d\Omega_k + d(V\rfloor\Omega_k) \equiv 0 \bmod \mathscr{I}$, and hence the conditions (1.26) imply satisfaction of the second of (1.27) for each value of the index k. Accordingly, the initial data conditions are satisfied if and only if the pair (V, ψ) satisfies the reduced initial data conditions

$$\psi^*\Omega_k = 0, \qquad \psi^*(V\rfloor\Omega_k) = 0 \tag{1.28}$$

for each value of the index k. This is a substantial reduction, as an elementary count will show.

2. THE EXTENSION OPERATOR ON THE LIE MANIFOLD OF A VECTOR FIELD

Classic isovector theory is concerned with situations in which the orbital parameter s of the flow associated with an isovector field is constant on a solution surface. In contrast, allowing s to vary over the solution manifold is precisely what is required in order to obtain an order-independent method of characteristics. We therefore need a new avenue of approach.

The graph space of the V-orbit of K takes the form

$$L = K \times \mathbb{R} \tag{2.1}$$

We refer to L as the *Lie manifold* of K and use a system of local coordinates $\{z^A, s | 1 \leq A \leq m\}$. The space K can be identified with the hypersurface $s = 0$ in L by the canonical inclusion

$$i: \quad K \to L \quad | \quad \{z^A\} \to \{z^A, 0\} \tag{2.2}$$

There is also the canonical projection

$$\pi: \quad L \to K \quad | \quad \{z^A, s\} \to \{z^A, 0\} \tag{2.3}$$

and hence any $\omega \in \Lambda(K)$ lifts to $\pi^*\omega \in \Lambda(L)$, which is independent of both s and ds. We may therefore consider any $\omega \in \Lambda(K)$ as lifted to an element of $\Lambda(L)$ by allowing π^* to act.

If $V \in T(K)$, we have the associated horizontal map

$$T_V: \quad L \to L \quad | \quad \{z^A, s\} \to \{\exp(s\mathscr{L}_V)\langle z^A\rangle, s\} \tag{2.4}$$

where we have used the fact that $Z^A(s) = \exp(s\mathscr{L}_V)\langle z^A\rangle$ is the operator representation of solutions of the orbital equations of the vector field V. It is then clear that

$$T_V(k) = \pi \circ (T_V|_{s=k}) \tag{2.5}$$

reproduces the automorphism $T_V(k)$ of K.

Let $\bar{d}$ denote the exterior derivative operator on $\Lambda(L)$. Since s as well as the z's can vary in L, we have

$$\bar{d} = d + ds \wedge \frac{\partial}{\partial s} \tag{2.6}$$

where

$$d = \bar{d}|_{s=k} \tag{2.7}$$

is the restriction of $\bar{d}$ to the hypersurface $s = k$. Noting that

$$d[\exp(s\mathscr{L}_V)\langle\omega\rangle] = \exp(s\mathscr{L}_V)\langle d\omega\rangle$$

$$ds \wedge \exp(s\mathscr{L}_V)\langle\omega\rangle = \exp(s\mathscr{L}_V)\langle ds \wedge \omega\rangle$$

it is easily seen that

$$\bar{d}[\exp(s\mathscr{L}_V)\langle z^A\rangle] = \exp(s\mathscr{L}_V)\langle dz^A + ds\ V \rfloor dz^A\rangle \tag{2.8}$$

Likewise, the identity

$$\mathscr{L}_{sV}\langle\omega\rangle = s\mathscr{L}_V\langle\omega\rangle + ds \wedge (V \rfloor \omega) \qquad \forall \omega \in \Lambda(K)$$

for any $V \in T(K)$ shows that

$$\exp(\mathscr{L}_{sV})\langle\omega\rangle = \exp(s\mathscr{L}_V)\langle\omega + ds \wedge (V \rfloor \omega)\rangle \tag{2.9}$$

A straightforward calculation then establishes the following basic result.

Theorem 2.1. If $\Omega \in \Lambda(K)$ and $V \in T(K)$, then

$$(\pi \circ T_V)^*\Omega = T_V^* \circ \pi^*\Omega = \exp(s\mathscr{L}_V)\langle\Omega + ds \wedge (V \rfloor \Omega)\rangle \tag{2.10}$$

This theorem provides the explicit new structure that is introduced by lifting considerations to the Lie manifold L where s is allowed to vary. In particular, if (2.10) is restricted to a hypersurface $s = k$, we obtain the results of classical isovector theory; namely

$$(T_V^* \circ \pi^*\Omega)|_{s=k} = \exp(k\mathscr{L}_V)\langle\Omega\rangle \tag{2.11}$$

The quantity on the right-hand side of (2.10) that is acted on by $\exp(s\mathscr{L}_V)$ is the extension of $\Omega \in \Lambda(K)$ to an element of $\Lambda(L)$ evaluated on the hypersurface $s = 0$ (i.e., on K). As such, it acts as the initial data on $s = 0$ for the orbital transport by V in the Lie manifold L.

Definition. Let $V \in TI$ and let Ω be a generic element of $\Lambda(K)$. The *V-extension operator* $E(V)$ defines a map from $\Lambda(K)$ to $\Lambda(L)$ by

$$E(V)\Omega = \Omega + ds \wedge (V \rfloor \Omega) \tag{2.12}$$

Lemma 2.1. The V-extension operator has the following properties:

$$E(V)(f\Omega_1+g\Omega_2)=fE(V)\Omega_1+gE(V)\Omega_2 \tag{2.13}$$

$$E(V)(\Omega_1\wedge\Omega_2)=[E(V)\Omega_1]\wedge[E(V)\Omega_2] \tag{2.14}$$

$$d(E(V)\Omega)=E(V)d\Omega+ds\wedge\mathscr{L}_V\Omega \tag{2.15}$$

$$E(V)(\mathscr{L}_V\Omega)=\mathscr{L}_V(E(V)\Omega) \tag{2.16}$$

for any $\Omega\in\Lambda(L)$ and any $V\in T(K)$, and we have the transport formula

$$T_V^*\circ\pi^*\Omega=\exp(s\mathscr{L}_V)\langle E(V)\Omega\rangle \tag{2.17}$$

for any $\Omega\in\Lambda(K)$.

Proof. Each of these results is obtained by direct calculation based on the definition of $E(V)$ and the standard definitions and properties of the exterior derivative, the inner product, and the Lie derivative. ■

An essential feature of the V-extension operator is the relation between isovectors of the fundamental ideal and isovectors of the V-extension of the fundamental ideal.

Definition. Let $\mathscr{I}=I\{\Omega_k, d\Omega_k|1\le k\le M\}$ be a differential ideal of $\Lambda(K)$. The *V-extension* $E(V)\mathscr{I}$ of the ideal $\mathscr{I}$ is the ideal generated by the V-extension of each of the generating forms of $\mathscr{I}$,

$$E(V)\mathscr{I}=I\{E(V)\Omega_k, E(V)\,d\Omega_k \;\mid\; 1\le k\le M\} \tag{2.18}$$

The collection of all isovectors of $E(V)\mathscr{I}$ is denoted by $TE(V)I$.

Remark. The ideal $E(V)\mathscr{I}$ is not a differential ideal because $E(V)$ and d do not commute in view of (2.15); that is, $E(V)\mathscr{I}\neq I\{E(V)\Omega_k, dE(V)\Omega_k|1\le k\le M\}$. This does not have serious consequences, however, as the proof of the next theorem shows.

Theorem 2.2. If V is an isovector of the differential ideal $\mathscr{I}$ of $\Lambda(K)$, then V is an isovector of the V-extension of $\mathscr{I}$,

$$V\in TI\Rightarrow V\in TE(V)I \tag{2.19}$$

Proof. Since d and $\mathscr{L}_V$ commute and $\mathscr{I}=I\{\Omega_k, d\Omega_k\}$ is a differential ideal, V is an isovector of $\mathscr{I}$ if and only if $\mathscr{L}_V\Omega_k\equiv 0 \bmod \mathscr{I}$, in which case we necessarily have $\mathscr{L}_V d\Omega_k\equiv 0 \bmod \mathscr{I}$. Equation (2.16) shows that $E(V)$ and $\mathscr{L}_V$ commute, and hence

$$\mathscr{L}_V E(V)\Omega_k=E(V)\mathscr{L}_V\Omega_k,\qquad \mathscr{L}_V E(V)\,d\Omega_k=E(V)\mathscr{L}_V d\Omega_k \tag{2.20}$$

Now, $\mathscr{L}_V\Omega_k$ and $\mathscr{L}_V\, d\Omega_k$ belong to $\mathscr{I}$ because $V \in TI$, and hence the action of $E(V)$ carries them into elements of $E(V)\mathscr{I}$. We therefore have

$$\mathscr{L}_V E(V)\Omega_k \equiv 0, \qquad \mathscr{L}_V E(V)\, d\Omega_k \equiv 0 \bmod E(V)\mathscr{I} \tag{2.21}$$

and hence V is an isovector of $E(V)\mathscr{I}$. ■

Let $\mathbb{R}^{p-1}$ have local coordinates $\{u^l | 1 \le l \le p-1\}$ and let $\mathbb{R}^p = \mathbb{R}^{p-1} \times \mathbb{R}$ have local coordinates $\{u^l; \tau | 1 \le l \le p-1\}$. These spaces will be used for parameter spaces in what follows. Let ψ be a map from an open set D_{p-1} of $\mathbb{R}^{p-1}$ into K that is quantified by

$$\psi \quad | \quad z^A = \psi^A(u^l) \tag{2.22}$$

Since K has been identified with the hypersurface $s = 0$ in L, the map ψ has the *cylindrical extension* $\hat{\psi}$ as a map from $D_{p-1} \times \mathbb{R} \subset \mathbb{R}^p$ into L that is quantified by

$$\hat{\psi} \quad | \quad z^A = \psi^A(u^l), \qquad s = \tau \tag{2.23}$$

We therefore have

$$\hat{\psi}^*\, dz^A = \psi^*\, dz^A, \qquad \hat{\psi}^*\, ds = d\tau \tag{2.24}$$

The workhorse of characteristic theory is the following result.

Theorem 2.3. Let V be an isovector of the fundamental ideal $\mathscr{I}$ and let $\mathbb{R}^p$ be a p-dimensional parameter space with local coordinates $\{u^l, \tau | 1 \le l \le p-1\}$. If

$$\psi \quad | \quad z^A = \psi^A(u^l) \tag{2.25}$$

is a map from an open subset D_{p-1} of $\mathbb{R}^{p-1}$ into K whose cylindrical extension is such that

$$\hat{\psi}^*(E(V)\mathscr{I}) = 0 \tag{2.26}$$

then there is a neighborhood $\mathbb{J}$ of $\tau = 0$ in $\mathbb{R}$ for which

$$\Psi = \pi \circ T_V \circ \hat{\psi} \quad | \quad z^A = Z^A(\psi^B(u^l), \tau) \tag{2.27}$$

is a map from the p-dimensional region $D_{p-1} \times \mathbb{J}$ of $\mathbb{R}^p$ into K that satisfies

$$\Psi^*\mathscr{I} = 0 \tag{2.28}$$

Proof. Standard manipulations, (2.27), and Theorem 2.1 give us

$$\Psi^*\mathscr{I} = \hat{\psi}^* \circ T_V^* \circ \pi^*\mathscr{I} = \hat{\psi}^* \exp(s\mathscr{L}_V)\langle E(V)\mathscr{I}\rangle \tag{2.29}$$

Since $V \in TI$ implies that V is an isovector of $E(V)\mathscr{I}$ by Theorem 2.2, and $\hat{\psi}^*$ annihilates $E(V)\mathscr{I}$ by hypothesis, it follows that $\hat{\psi}^*$ annihilates $\exp(s\mathscr{L}_V)\langle E(V)\mathscr{I}\rangle$. The result then follows from (2.29). ■

All that now remains is to obtain explicit characterization of the conditions imposed by the requirements $\hat{\psi}^*(E(V)\mathscr{I}) = 0$.

Lemma 2.2. The map $\hat{\psi}^*$ annihilates $E(V)\mathscr{I}$ with

$$\mathscr{I} = I\{\Omega_k, d\Omega_k | 1 \le k \le M\}$$

if and only if

$$\psi^*\Omega_k = 0, \qquad \psi^*(V|\Omega_k) = 0, \qquad 1 \le k \le M \tag{2.30}$$

Proof. Since $\hat{\psi}^*$ satisfies (2.24) by construction, a direct calculation based on the definition of $E(V)$ gives

$$\hat{\psi}^* E(V)\omega = \psi^*\omega + d\tau \wedge \psi^*(V \rfloor \omega) \tag{2.31}$$

for any $\omega \in \Lambda(K)$. Now, $E(V)\mathscr{I} = I\{E(V)\Omega_k, E(V)\, d\Omega_k | 1 \le k \le M\}$, and hence $\hat{\psi}^*$ will annihilate $E(V)\mathscr{I}$ if and only if it annihilates each of the generators of $E(V)\mathscr{I}$. When (2.31) is used, this means that we must have

$$\psi^*\Omega_k + d\tau \wedge \psi^*(V \rfloor \Omega_k) = 0 \tag{2.32}$$

$$\psi^*\, d\Omega_k + d\tau \wedge \psi^*(V \rfloor d\Omega_k) = 0 \tag{2.33}$$

Now, $d\tau$, du^1, $du^2, \ldots, du^{p-1}$ are independent 1-forms, while the domain of the map ψ is the parameter space D_{p-1} with local coordinates $u^1, u^2, \ldots, u^{p-1}$. It thus follows that (2.32) and (2.33) can be satisfied if and only if

$$\psi^*\Omega_k = 0, \qquad \psi^*(V \rfloor \Omega_k) = 0$$

and

$$\psi^*\, d\Omega_k = 0, \qquad \psi^*(V \rfloor d\Omega_k) = 0$$

are satisfied. It is then trivial to see that satisfaction of the second set is implied by satisfaction of the first set when we use the fact that V is an isovector of the fundamental ideal $\mathscr{I}$. ■

If ψ were a map of rank $p-1$ and the isovector V were transverse to the range of ψ in K, then Ψ would have rank p for τ in a neighborhood of $\tau = 0$. In this event, Theorem 2.3 shows how to construct an annihilating map Ψ of the ideal $\mathscr{I}$ of rank p from an annihilating map ψ of rank $p-1$. Under these circumstances, ψ may be viewed as defining an initial data manifold of dimension $p-1$ for the construction of an annihilating map of the ideal $\mathscr{I}$ by transport of these initial data along the orbits of the isovector field V of $\mathscr{I}$. Theorem 2.3 thus provides a natural basis for a strong dimension reduction procedure for PDE that can be characterized by a differential ideal.

3. CHARACTERISTIC SOLUTIONS OF PDE THAT ARE CHARACTERIZED BY A DIFFERENTIAL IDEAL

Partial differential equations that can be characterized by a differential ideal $\mathscr{I}$ of $\Lambda(K)$ have been discussed in Section 1. The essential idea here is to concentrate on the collection of all solution maps

$$S = \{\Phi\colon M_n \to K \quad | \quad \Phi^*\mu \neq 0,\ \Phi^*\mathscr{I} = 0\} \tag{3.1}$$

The results established in Theorem 2.3 show us how to construct a map Ψ that annihilates the fundamental ideal $\mathscr{I}$. We therefore need to determine the conditions under which a map of the form given by (2.27) will satisfy the further condition $\Psi^*\mu \neq 0$.

Theorem 3.1. Let V be an isovector of the fundamental ideal $\mathscr{I}$ and let ψ be a map from D_{n-1} into K that satisfies the transversality condition

$$\psi^*(V \rfloor \mu) \neq 0 \tag{3.2}$$

Then ψ is or rank $n-1$,

$$\Psi = \pi \circ T_V \circ \hat{\psi} \tag{3.3}$$

is of rank n, and

$$\Psi^*\mu \neq 0 \tag{3.4}$$

on $D_{n-1} \times \mathbb{J}$ for some neighborhood $\mathbb{J}$ of $\tau = 0$.

Proof. Since $V \rfloor \mu$ is an $(n-1)$-form on K and ψ has an $(n-1)$-dimensional domain D_{n-1}, satisfaction of the transversality condition (3.2) implies that ψ is of maximal rank $n-1$. On the other hand, (3.3), (2.17), and (2.27) give

$$\Psi^*\mu = \hat{\psi}^* \exp(s\mathscr{L}_V)\langle \mu + ds \wedge (V \rfloor \mu)\rangle \tag{3.5}$$

Noting that $\exp(s\mathscr{L}_V)\langle \mu \rangle$ is an n-form, that ψ is of rank $n-1$, and $\hat{\psi}^*\mu = \psi^*\mu = 0$, one finds that (3.5) reduces to

$$\Psi^*\mu = \hat{\psi}^* \exp(s\mathscr{L}_V)\langle ds \wedge (V \rfloor \mu)\rangle \tag{3.6}$$

The continuity of the right-hand side of (3.6) then shows that satisfaction of the transversality condition implies the existence of a neighborhood $\mathbb{J}$ of $\tau = 0$ such that $\Psi^*\mu \neq 0$, and hence Ψ has rank n on $D_{n-1} \times \mathbb{J}$. ■

It is now simply a matter of combining Theorem 2.3, Lemma 2.2, and Theorem 3.1 in order to obtain the following result.

Theorem 3.2. Let K be an m-dimensional space with local coordinates $\{z^A | 1 \leq A \leq m\}$ and let $\mathscr{I} = I\{\Omega_k, d\Omega_k | 1 \leq k \leq M\}$ be a differential ideal of

$\Lambda(K)$ that characterizes a system of PDE. If V is an isovector of $\mathscr{I}$ and ψ is a map from D_{n-1} into K such that

$$\psi^*(V \rfloor \mu) \neq 0 \tag{3.7}$$

$$\psi^* \Omega_k = 0, \qquad 1 \leq k \leq M \tag{3.8}$$

$$\psi^*(V \rfloor \Omega_k) = 0, \qquad 1 \leq k \leq M \tag{3.9}$$

then

$$\Psi = \pi \circ T_V \circ \hat{\psi} \tag{3.10}$$

is a map from $D_{n-1} \times \mathbb{J} \subset \mathbb{R}^n$ into K that solves the given system of PDE (i.e., $\Psi \in S$).

If

$$V = v^A(z^B)\partial_A \tag{3.11}$$

and

$$\psi: \quad D_{n-1} \to K | z^A = \psi^A(u^I) \tag{3.12}$$

satisfy the hypotheses of Theorem 3.2, then the solving map Ψ is given by

$$\Psi: \quad D_{n-1} \times \mathbb{J} \to K \quad | \quad z^A = Z^A(\psi^B(u^I); \tau) \tag{3.13}$$

where the Z's are obtained by solving the initial value problem

$$\frac{dZ^A}{d\tau} = v^A(Z^B), \qquad Z^A(0) = \psi^A(u^I) \tag{3.14}$$

Equations (3.14) may be viewed as the equations for the *characteristics* associated with the pair (V, ψ) that satisfy the transversality condition (3.7). In like manner, the image of D_{n-1} in K will be referred to as the *initial data manifold*, while (3.8) and (3.9) will be referred to as the *initial data constraints*. Theorem 3.2 can thus be interpreted in the following manner.

If V is an isovector of the fundamental ideal $\mathscr{I}$ and the pair (V, ψ) satisfies the transversality condition (3.7), then the given system of PDE admits a characteristic solution Ψ of the form (3.13) provided the initial data satisfy the initial data constraints (3.8) and (3.9) on the initial data manifold.

A characteristic solution of a system of PDE that is generated by an isovector V of the fundamental ideal $\mathscr{I}$ necessarily satisfies a system of additional conditions.

Theorem 3.3. If Ψ is a characteristic solution of a system of PDE that is generated by an isovector V of the fundamental ideal $\mathscr{I} = I\{\Omega_k, d\Omega_k | 1 \leq k \leq M\}$, then Ψ satisfies the conditions

$$\Psi^*(V \rfloor \Omega_k) = 0, \qquad 1 \leq k \leq M \tag{3.15}$$

throughout the domain of Ψ. Characteristic solutions of the system of PDE are thus restricted (as opposed to general) solutions that satisfy a specific set of additional conditions except in those cases where $V \rfloor \mathscr{I} \subset \mathscr{I}$.

Proof. An obvious calculation shows that

$$E(V)(V \rfloor \Omega_k) = V \rfloor \Omega_k \tag{3.16}$$

and hence

$$\Psi^*(V \rfloor \Omega_k) = \hat{\psi}^* \exp(s\mathscr{L}_V)\langle V \rfloor \Omega_k \rangle \tag{3.17}$$

Now, $\mathscr{L}_V(V \rfloor \Omega_k) = V \rfloor (\mathscr{L}_V \Omega_k)$ because $\mathscr{L}_V V = 0$, and $\mathscr{L}_V \Omega_k \equiv 0 \bmod \mathscr{I}$ because $V \in TI$. Hence $\mathscr{L}_V(V \rfloor \Omega_k)$ is contained in the ideal generated by $\{\Omega_k, V \rfloor \Omega_k | 1 \leq k \leq M\}$. Accordingly, $\exp(s\mathscr{L}_V)\langle V \rfloor \Omega_k \rangle$ is contained in the ideal generated by $\{\Omega_k, V \rfloor \Omega_k\}$. Satisfaction of the initial data constraints (3.7) and (3.8) thus implies that the right-hand side of (3.17) necessarily vanishes for a characteristic solution generated by (V, ψ), and hence (3.15) are satisfied. These conditions are explicit new conditions that the characteristic solution will satisfy unless $V \rfloor \mathscr{I} \subset \mathscr{I}$, in which case $\Psi^* \mathscr{I} = 0$ will imply $\Psi^*(V \rfloor \mathscr{I}) = 0$. ■

Remark. An easy calculation shows that the pullback by a solution map of the higher-order contact 1-forms gives

$$\Psi^*(V \rfloor C^\alpha_i) = \frac{d}{dx^i} \Psi^*(V \rfloor C^\alpha), \qquad \Psi^*(V \rfloor C^\alpha_{ij}) = \frac{d^2}{dx^i\, dx^j} \Psi^*(V \rfloor C^\alpha) \tag{3.18}$$

This shows that the only independent constraints that come from any collection of contact 1-forms of any finite order are those that come from the contact forms of first order; namely

$$\Psi^*(V \rfloor C^\alpha) = 0 \tag{3.19}$$

This observation will be of importance in correlating characteristic solutions with group invariant (similarity) solutions of PDE in Section 7.

4. PROBLEMS WITH FIRST-ORDER CONTACT 1-FORMS

Most of the remainder of this paper will be confined to situations in which only first-order contact forms appear in the fundamental ideal $\mathscr{I}$. The space K is therefore of dimension $m = n + N + nN$ with local coordinates $\{x^i, q^\alpha, y^\alpha_i | 1 \leq i \leq n, 1 \leq \alpha \leq N\}$. The ideal $\mathscr{I}$ thus has the form

$$\mathscr{I} = I\{C^\alpha, dC^\alpha, H^a, dH^a \quad | \quad 1 \leq \alpha \leq N, 1 \leq a \leq r\} \tag{4.1}$$

where

$$C^\alpha = dq^\alpha - y^\alpha_k\, dx^k \tag{4.2}$$

and the H's are elements of $\Lambda(K)$ that describe the PDE under study. The contact ideal

$$\mathscr{C} = I\{C^\alpha, dC^\alpha \quad | \quad 1 \leq \alpha \leq N\} \tag{4.3}$$

is clearly a subideal of $\mathscr{I}$, and the structure of the isovectors of $\mathscr{C}$ is known from previous work (Edelen, 1985, Theorem 6-4.1). The following notation will be used:

$$\partial_i = \partial/\partial x^i, \qquad \partial_\alpha = \partial/\partial q^\alpha, \qquad \partial^i_\alpha = \partial/\partial y^\alpha_i$$

Theorem 4.1. A vector field

$$V = v^i\partial_i + \bar{v}^\alpha\partial_\alpha + v^\alpha_i\partial^i_\alpha \tag{4.4}$$

is an isovector of the contact ideal if and only if

$$v^\alpha_i = Z_i\langle V \rfloor C^\alpha\rangle = Z_i\langle \bar{v}^\alpha - y^\alpha_k v^k\rangle \tag{4.5}$$

$$\mathscr{L}_V C^\alpha = \partial_\beta\langle V \rfloor C^\alpha\rangle C^\beta \tag{4.6}$$

where Z_i is the linear differential operator

$$Z_i = \partial_i + y^\beta_i\partial_\beta \tag{4.7}$$

and for $N > 1$

$$v^i = f^i(x^j, q^\beta), \qquad \bar{v}^\alpha = f^\alpha(x^j, q^\beta) \tag{4.8}$$

while for $N = 1$

$$v^i = \partial^i_1\langle\eta(x^j, q^1, y^1_j)\rangle, \qquad \bar{v}^1 = (y^i_1\partial^1_i - 1)\langle\eta\rangle \tag{4.9}$$

in which case

$$V \rfloor C^1 = -\eta(x^j, q^1, y^1_j) \tag{4.10}$$

A comparison of these results with those reported by Olver (1986) leads to the following interrelations. An isovector V of the contact ideal is the *first prolongation* of a vector field

$$V_G = v^i\partial_i + \bar{v}^\alpha\partial_\alpha$$

on graph space for $N > 1$; that is, $V = \mathbf{pr}^1(V_G)$ for $N > 1$. In fact, TI and $\mathbf{pr}^1(T(G))$ coincide when $N > 1$. On the other hand, for $N = 1$, an isovector of the contact ideal is a first prolongation of a vector field on graph space only when the function $\eta(x^j, q^1, y^1_j)$ that appears in (4.9) and (4.10) is an affine function of the y's. Situations with $N = 1$ and η nonlinear in the y's are thus precluded if the analysis is restricted to prolongations via the standard jet bundle formulation! The possible drastic differences between isovector methods and prolongation methods in cases with $N = 1$ suggests that we look at specific cases for $N = 1$ for which η is not affine in the y's.

5. PROBLEMS WITH ONLY ONE DEPENDENT VARIABLE

5.1. First-Order Equations

I first consider a single first-order PDE

$$h(x^i, U(x^j), \partial_i U(x^j)) = 0 \tag{5.1}$$

in a single unknown $U(x^j)$, so that $N = 1$ and $n > 1$. In order to simplify the notation, I will use local coordinates $\{x^j, q, y_i \mid 1 \leq i \leq n\}$ on K so that the single contact 1-form becomes

$$C^1 = dq - y_i\, dx^i \tag{5.2}$$

The fundamental ideal in this case is

$$\mathscr{I} = I\{C^1, dC^1, H, dH\} \tag{5.3}$$

where H is the element of $\Lambda^0(K)$ that is defined by

$$H = h(x^i, q, y_i) \tag{5.4}$$

Noting that

$$\mathscr{L}_V(V \rfloor C^1) = V \rfloor \mathscr{L}_V C^1 = \partial_q \langle V \rfloor C^1 \rangle V \rfloor C^1 \tag{5.5}$$

for any isovector of the contact ideal, by Theorem 4.1, it follows that

$$\eta(x^i, q, y_i) = H(x^i, q, y_i) = -V \rfloor C^1 \tag{5.6}$$

defines an isovector

$$V = \partial^i \langle H \rangle \partial_i + (y_i \partial^i - 1)\langle H \rangle \partial_q - Z_i \langle H \rangle \partial^i \tag{5.7}$$

of the fundamental ideal

$$\mathscr{I} = I\{C^1, dC^1, H, dH\} = I\{C^1, dC^1, -V \rfloor C^1, -d(V \rfloor C^1)\}.$$

We are therefore in position to apply Theorem 3.2 where $V \rfloor \mathscr{I} \subset \mathscr{I}$.

Since we are dealing with a PDE, the function $H(x^i, q, y_i)$ cannot be independent of all of the y's. Hence, $\partial^i \langle H \rangle$ cannot vanish throughout K for all values of the index i. Accordingly,

$$V \rfloor \mu = v^i \mu_i = \partial^i \langle H \rangle \mu_i \tag{5.8}$$

is an $(n-1)$-form on K that does not vanish throughout K. Let

$$\psi\colon \quad D_{n-1} \to K \quad | \quad x^i = \psi^i(u^l), \qquad q = \psi(u^l), \qquad y_i = \psi_i(u^l) \tag{5.9}$$

be a candidate initial data map for the PDE $h = 0$. The transversality condition (3.7) requires that the initial data be chosen so that

$$\psi^*(V \rfloor \mu) = \psi^*(\partial^i \langle H \rangle)\psi^* \mu_i \neq 0 \tag{5.10}$$

When (5.10) is satisfied, we have $\psi^* \mu_i \neq 0$ for at least one value of the index i. Hence ψ is of rank $n-1$ and

$$\operatorname{rank}(\partial \psi^i / \partial u^l) = n-1 \tag{5.11}$$

on D_{n-1}. Now, for the case at hand, we have $V \rfloor \mathscr{I} \subset \mathscr{I}$, and hence the initial data conditions (3.8) and (3.9) reduce to

$$\psi^* C^1 = \left(\frac{\partial \psi}{\partial u^l} - \psi_j \frac{\partial \psi^j}{\partial u^l} \right) du^l = 0 \tag{5.12}$$

$$\psi^*(V \rfloor C^1) = -H(\psi^i, \psi, \psi_i) = 0 \tag{5.13}$$

Equations (5.12) may be written as a system of $n-1$ affine equations

$$\psi_i \frac{\partial \psi^i}{\partial u^l} = \frac{\partial \psi}{\partial u^l} \tag{5.14}$$

for the determination of the n functions $\psi_i(u^l)$. Since the coefficient matrix of this system has rank $n-1$, the system (5.14) will always have a one-dimensional affine set of solutions. When these solutions are substituted into the remaining initial data constraint equation (5.13), a single equation is obtained for the determination of the required elements of the affine set of solutions of (5.14), and this single equation will have a solution by the implicit function theorem as a direct consequence of the satisfaction of the transversality condition. Each such solution will thus determine the n functions $\psi_i(u^l)$ in terms of the functions $\psi^i(u^l)$, $\psi(u^l)$, and their first derivatives.

Satisfaction of the conditions just stated gives the initial data $\{\psi^i(u^l), \psi(u^l), \psi_i(u^l) | 1 \leq i \leq n\}$. Theorem 3.2 then shows that an implicitly defined solution of the given PDE is obtained by solving the initial value problem

$$\frac{dX^i}{d\tau} = \frac{\partial \bar{H}}{\partial Y_i}, \qquad X^i(0) = \psi^i(u^l) \tag{5.15}$$

$$\frac{dQ}{d\tau} = Y_j \frac{\partial \bar{H}}{\partial Y_j} - \bar{H}, \qquad Q(0) = \psi(u^l) \tag{5.16}$$

$$\frac{dY_i}{d\tau} = -\frac{\partial \bar{H}}{\partial X^i} - Y_i \frac{\partial \bar{H}}{\partial Q}, \qquad Y_i(0) = \psi_i(u^l) \tag{5.17}$$

for the characteristics. Here $\bar{H}$ is defined in terms of H by $\bar{H} = H(X^i, Q, Y_i)$. If the solution to this initial value problem is written in the form (notice that we have to solve the Y equations as well as those for the X's and for Q)

$$X^i = \hat{X}(\psi^j(u^l), \psi(u^l), \psi_i(u^l); \tau) \tag{5.18}$$

$$Q = \hat{Q}(\psi^i(u^{\mathscr{S}}), \psi(u^l), \psi_i(u^l); \tau) \tag{5.19}$$

then the original PDE $h = 0$ has the implicitly defined solution

$$x^i = \hat{X}^i, \qquad U = \hat{Q} \tag{5.20}$$

This process is, however, exactly the standard method of characteristics (Duff, 1956; Courant and Hilbert, 1937) for the partial differential equation

$$h(x^i, U(x^j), \partial_i U(x^j)) = 0$$

Theorem 3.2 is thus in agreement with the classical theory of characteristics for a first-order PDE with a single unknown. The reader might be worried about the additional term $-\bar{H}$ on the right-hand side of (5.16). It is easily shown, however, that

$$\frac{d\bar{H}}{d\tau} = -\bar{H}\frac{\partial \bar{H}}{\partial Q}$$

and hence the initial data $\bar{H}(0) = 0$ implies that $\bar{H}$ will vanish on any orbit.

Although Theorem 3.3 still holds, a characteristic solution does not satisfy additional constraints, because $H = -V \rfloor C^1$ (i.e., $V \rfloor \mathcal{J} \subset \mathcal{J}$). The collection of all characteristic solutions of the PDE $h = 0$ thus contains all smooth solutions of this PDE for which the initial data satisfy the transversality condition.

As an example, let us look at the problem with $n = 2$ with independent variables $\{x, t\}$ and

$$h(x, t, U, \partial_x U, \partial_t U) = \partial_x U\, \partial_t U - 4U = 0$$

In this instance, we have $H = y_x y_t - 4q$, and $\eta = H$ generates an isovector of the fundamental ideal $I\{C^1, dC^1, H, dH\}$. This isovector gives $-V \rfloor C^1 = \eta = H$, and hence $V \rfloor \mathcal{J} \subset \mathcal{J}$. Hence, characteristic solutions generated by this isovector do not satisfy additional differential constraints. If we take the initial data map of the form

$$\psi: \quad \mathbb{R} \to K \quad | \quad x = u, \quad t = 0, \quad q = \alpha(u), \quad y_x = \psi_x(u), \quad y_t = \psi_t(u)$$

then the transversality condition is satisfied provided $\alpha' = d\alpha/du \neq 0$, while the initial data conditions give the evaluations

$$\psi_x = \alpha', \qquad \psi_t = 4\alpha/\alpha'$$

An integration of the orbital equations

$$\frac{dX}{dT} = Y_t, \qquad \frac{dT}{d\tau} = Y_x, \qquad \frac{dQ}{d\tau} = Y_x Y_t + 4Q$$

$$\frac{dY_x}{d\tau} = 4Y_x, \qquad \frac{dY_t}{d\tau} = 4Y_t$$

with the initial data determined by the map ψ gives us the implicitly defined solution

$$x = u + \frac{\alpha(u)}{\alpha'(u)}(e^{4\tau} - 1), \qquad t = \tfrac{1}{4}\alpha'(u)(e^{4\tau} - 1), \qquad U = \alpha(u)e^{8\tau}$$

for any choice of the initial data $U(x, 0) = \alpha(x)$ with $\alpha'(x) \neq 0$.

5.2. Second-Order Equations

A second-order PDE in a single unknown can often be represented in terms of the fundamental ideal

$$\mathcal{I} = I\{C^1, dC^1, H^1, dH^1\} \tag{5.21}$$

where H^1 is a 2-form on K. The best way to proceed here is with an example. For simplicity, we take $n = 2$ and use $\{x, t, q, y_x, y_t\}$ as a system of local coordinates on K. The field equation to be studied is characterized by the 2-form

$$H^1 = d\alpha_1 \wedge d\alpha_2 \tag{5.22}$$

with

$$\alpha_1 = x^2 + (y_t)^2, \qquad \alpha_2 = t^2 + (y_x)^2 \tag{5.23}$$

If $\Phi\colon (x, t) \to K$ is a solving map of the fundamental ideal with the presentation

$$x = x, \qquad t = t, \qquad q = \phi(x, t), \qquad y_x = \partial_x \phi, \qquad y_t = \partial_t \phi \tag{5.24}$$

then (5.22) and (5.23) give the Monge–Ampere-like nonlinear PDE

$$(x + \partial_t\phi\ \partial_x\partial_t\phi)(t + \partial_x\phi\ \partial_x\partial_t\phi) = \partial_x\phi\ \partial_t\phi\ \partial_x\partial_x\phi\partial_t\partial_t\phi \tag{5.25}$$

A direct calculation shows that an isovector of the fundamental ideal $\mathcal{I}$ is generated by

$$\eta = y_x y_t + xt \tag{5.26}$$

in accordance with Theorem 4.1 with $N = 1$;

$$V = y_t\partial_x + y_x\partial_t + (y_x y_t - xt)\partial_q - t\,\partial_{y_x} - x\,\partial_{y_t} \tag{5.27}$$

Let ψ be an initial data map from $D_1 \subset \mathbb{R}$ into K that is quantified by

$$x = u, \qquad t = 1, \qquad q = f(u), \qquad y_x = \psi_1(u), \qquad y_t = \psi_2(u) \tag{5.28}$$

and let f' denote the derivative of $f(u)$ with respect to u. We then have $\psi^* C^1 = (f' - \psi_1)\,du$, and hence $\psi^* C^1 = 0$ gives

$$\psi_1(u) = f'(u) \tag{5.29}$$

Since $V \rfloor \mu = y_t\, dt - y_x\, dx$, the transversality condition $\psi^*(V \rfloor \mu) \neq 0$ will be satisfied for all $u \in D_1$ for which

$$f'(u) \neq 0 \tag{5.30}$$

Next, we note that

$$V \rfloor C^1 = -y_x y_t - xt = -\eta \tag{5.31}$$

and hence $\psi^*(V \rfloor C^1) = 0$ gives the determination

$$\psi_2(u) = -\frac{u}{f'(u)} \tag{5.32}$$

Since H^1 is a 2-form and ψ has rank 1, $\psi^* H^1 = 0$. On the other hand, a direct calculation shows that $V \rfloor H^1 = 0$ (i.e., H^1 is an absolute invariant of the flow generated by V), and hence $\psi^*(V \rfloor H^1) \equiv 0$; the reduced field equation is satisfied identically. There are thus no further restrictions that the initial data must satisfy for this problem. In the general situation, $V \rfloor H^1 \neq 0$, there would be an additional condition (the reduced field equation) that would involve both first and second derivatives of the function $f(u)$, and this additional condition would have to be solved in order to determine the admissible functions $f(u)$.

What now remains is to solve the orbital equations

$$\begin{gathered} \frac{dX}{d\tau} = Y_t, \qquad \frac{dT}{d\tau} = Y_x, \qquad \frac{dQ}{d\tau} = Y_x Y_t - XT \\ \frac{dY_x}{d\tau} = -T, \qquad \frac{dY_t}{d\tau} = -X \end{gathered} \tag{5.33}$$

subject to the initial data

$$\begin{gathered} X(0) = u, \qquad T(0) = 1, \qquad Q(0) = f(u) \\ Y_x(0) = f'(u), \qquad Y_t(0) = -\frac{u}{f'(u)} \end{gathered} \tag{5.34}$$

Although the Y's are now explicitly included since the isovector is not a prolongation of a vector field on graph space, solutions of the given system of PDE are derived implicitly by setting $x = X(u; \tau)$, $t = T(u; \tau)$, $\phi = Q(u; \tau)$. We therefore have a characteristic solution

$$x = u\cos(\tau) - \frac{u}{f'(u)}\sin(\tau), \qquad t = \cos(\tau) + f'(u)\sin(\tau) \tag{5.35}$$

$$\phi = f(u) - u\sin(2\tau) - \frac{u}{2}\left[f'(u) - \frac{1}{f'(u)}\right][\cos(2\tau) - 1] \tag{5.36}$$

for any choice of the initial data

$$\phi(x, 1) = f(x) \tag{5.37}$$

such that $f'(x) \neq 0$ on the initial data line $t = 1$. This characteristic solution is single-valued for all values of τ in a neighborhood $\mathbb{J}$ of $\tau = 1$ where $\partial(x, t)/\partial(u, \tau) \neq 0$.

Although the initial data $f(u)$ can be arbitrary subject to the condition $f'(u) \neq 0$, the given PDE has many other solutions that are not characteristic solutions. The reason for this is that $V \rfloor \mathscr{I}$ is not contained in $\mathscr{I}$, and hence any characteristic solution satisfies the constraint

$$\Psi^*(V \rfloor C^1) = 0 \tag{5.38}$$

Use of the evaluation (5.31) shows that the characteristic solution obtained from the isovector V necessarily satisfies the differential constraint

$$\partial_x \phi \, \partial_t \phi + xt = 0 \tag{5.39}$$

no matter how we choose the initial data function $f(u)$. Indeed, since we are solving the second-order PDE (5.25), the fact that we are only allowed to assign the initial data $\phi(x, 1) = f(x)$ indicates that the solution is not general. That this is the case is also in evidence from the fact that (5.34) gives the initial velocity data

$$\partial_t \phi(x, 1) = -\frac{x}{f'(x)} \tag{5.40}$$

in agreement with (5.39), because $\partial_x \phi(x, 1) = f'(x)$. The important thing to note is the relative ease with which we have been able to construct characteristic solutions to the complicated PDE (5.25). This is indicative of the utility of this order-independent method of characteristics even though it does not give us all solutions of the problem.

6. PROBLEMS WITH SEVERAL DEPENDENT VARIABLES

The fundamental ideal for problems with $N > 1$ has the generic form

$$\mathscr{I} = I\{C^\alpha, dC^\alpha, C_i^\alpha, dC_i^\alpha, \ldots, H^a, dH^a \mid 1 \leq \alpha \leq N, 1 \leq i \leq n, 1 \leq a \leq r\} \tag{6.1}$$

Accordingly, Theorem 4.1 and the results given in Olver (1986) show that any isovector of $\mathscr{I}$ is a prolongation of a vector field

$$V_G = v^i(x^j, q^\beta)\partial_i + \bar{v}^\alpha(x^j, q^\beta)\partial_a \tag{6.2}$$

on graph space. In particular, we have

$$v_i^\alpha = Z_i\langle v^\alpha - y_j^\alpha v^j\rangle, \ldots \tag{6.3}$$

In order to satisfy the hypotheses of Theorem 3.2, V and the initial data map

$$\psi: D_{n-1} \to K \quad | \quad x^i = \psi^i(u^l), \qquad q^\alpha = \hat{\psi}^\alpha(u^l), \qquad y_i^\alpha = \psi_i^\alpha(u^l), \ldots \tag{6.4}$$

must satisfy the transversality condition $\psi^*(V \rfloor \mu) = 0$; that is,

$$\psi^*(v^i(x^j, q^\beta))\psi^*(\mu_i) \neq 0 \tag{6.5}$$

We therefore have the following elementary result.

Lemma 6.1. A system of PDE with $N > 1$ has a solution by the method of characteristics only if the fundamental ideal admits an isovector V for which at least one of the coefficient functions $v^i(x^j, q^\beta)$ is nonzero in some open subset of K for at least one value of the index i.

I will assume that the conditions of Lemma 6.1 are met since I am only interested in those isovectors for which the method of characteristics is applicable. In particular, one must exclude isovectors for internal symmetries because such isovectors have $v^i = 0$ for all values of the index i.

In order to simplify the notation, put

$$V^i = \psi^* v^i(x^j, q^\beta) = v^i(\psi^j(u^l), \hat{\psi}^\beta(u^l)) \tag{6.6}$$

$$\hat{V}^\alpha = \psi^* \hat{v}^\alpha(x^j, q^\beta) = \hat{v}^\alpha(\psi^j(u^l), \hat{\psi}^\beta(u^l)), \ldots \tag{6.7}$$

The following result is then easily obtained from (6.5) and the column expansion formula for a determinant.

Lemma 6.2. If V is an isovector of the fundamental ideal and ψ is an initial data map for $N > 1$, then

$$\psi^*(V \rfloor \mu) = (-1)^{1+n} \det(\mathbf{M})\, du^1 \wedge du^2 \wedge \cdots \wedge du^{n-1} \tag{6.8}$$

where $\mathbf{M}$ is the $n \times n$ matrix

$$\mathbf{M} = \left(\left(\frac{\partial \psi^i}{\partial u^l}, V^i\right)\right) \tag{6.9}$$

The transversality condition is therefore satisfied if and only if

$$\det(\mathbf{M}) \neq 0 \tag{6.10}$$

is satisfied throughout the domain D_{n-1} of the initial data map ψ.

Theorem 6.1. If the pair (V, ψ) satisfy the transversality condition, then the initial data constraints

$$\psi^* C^\alpha = 0, \qquad \psi^*(V \rfloor C^\alpha) = 0 \tag{6.11}$$

$$\psi^* C_i^\alpha = 0, \qquad \psi^*(V \rfloor C_i^\alpha) = 0, \ldots \tag{6.12}$$

uniquely determine the initial data $\psi_i^\alpha, \psi_{ij}^\alpha, \ldots$ in terms of the initial data ψ^i, ψ^α, and their partial derivatives with respect to the parameters u^l.

Proof. An elementary calculation and (6.4) show that

$$\psi^* C^\alpha = 0, \qquad \psi^*(V \rfloor C^\alpha) = 0$$

give the relations

$$\psi_m^\alpha \frac{\partial \psi^m}{\partial u^l} = \frac{\partial \psi^\alpha}{\partial u^l}, \qquad \psi_m^\alpha V^m = V^\alpha \tag{6.13}$$

This is an affine system of equations for the determination of the row matrix $[\psi_m^\alpha | 1 \le m \le n]$ for each value of the index α because the functions V^m and V^α are independent of ψ_m^α. Noting that the coefficient matrix of this system is the matrix **M**, which is nonsingular whenever the transversality condition is satisfied, we find that the system (6.13) serves to determine all of the initial data functions ψ_i^α. Similarly,

$$\psi^* C_i^\alpha = 0, \qquad \psi^*(V \rfloor C_i^\alpha) = 0$$

give the relations

$$\psi_{im}^\alpha \frac{\partial \psi^m}{\partial u^l} = \frac{\partial \psi_i^\alpha}{\partial u^l}, \qquad \psi_{im}^\alpha V^m = V_i^\alpha \tag{6.14}$$

Since the coefficient matrix for this affine system is also **M**, and the same is true for all initial data conditions that come from contact forms of higher order, the result is established. ■

Theorem 6.2. A system of PDE with $N > 1$ that is characterized by a fundamental ideal $\mathcal{I}$ has a solution map $\Psi = \hat{\psi} \circ T_V \circ \pi$ given by the method of characteristics if and only if $\mathcal{I}$ admits an isovector V and a mapping ψ: $D_{n-1} \to K$ that satisfy the transversality condition

$$\psi^*(V \rfloor \mu) \neq 0 \tag{6.15}$$

and the reduced field equations

$$\psi^* H^a = 0, \qquad \psi^*(V \rfloor H^a) = 0, \qquad 1 \le a \le r \tag{6.16}$$

throughout the domain of ψ.

Proof. Theorem 6.1 shows that all of the initial data constraints that come from the contact ideal can be satisfied whenever the transversality condition is satisfied. Accordingly, Theorem 3.2 shows that we have a characteristic solution whenever the remaining initial data constraints (6.16) are satisfied. ■

Remark. Most problems have H's that are either 0-forms (when the order of the contact 1-forms is equal to the order of the highest derivative)

or n-forms (when the order of the contact 1-forms is one less than the order of the highest derivative). If the H's are 0-forms, $V \rfloor H^a \equiv 0$ and (6.16) reduce to the r conditions $\psi^* H^a = 0$. If the H's are n-forms, $\psi^* H^a \equiv 0$ because ψ has rank $n-1$, and hence (6.16) reduce to the r conditions $\psi^*(V \rfloor H^a) = 0$. The number of original equations and the number of reduced field equations are thus the same.

Systems of quasilinear first-order PDE with the same principal part,

$$f^i(x^j, \phi^\beta) \frac{\partial \phi^\alpha}{\partial x^i} - f^\alpha(x^j, \phi^\beta) = 0 \tag{6.17}$$

are known to be solvable by the classical method of characteristics (Duff, 1956). Such systems can be characterized by the fundamental ideal

$$\mathcal{I} = I\{C^\alpha, dC^\alpha, H^\alpha, dH^\alpha\}$$

on a contact manifold K with local coordinates $\{x^i, q^\alpha, y_i^\alpha | 1 \le i \le n, 1 \le \alpha \le N\}$ by simply taking $H^\alpha = f^i(x^j, q^\beta) y_i^\alpha - f^\alpha(x^j, q^\beta)$. Now, an isovector of $\mathcal{I}$ is generated by

$$V_G = f^i(x^j, q^\beta)\partial_i + f^\alpha(x^j, q^\beta)\partial_\alpha \tag{6.18}$$

For this isovector, $V \rfloor C^\alpha = -H^\alpha$ and hence $V \rfloor \mathcal{I} \subset \mathcal{I}$. The method of characteristics presented here can thus be applied to such systems, and we will obtain general solutions because $V \rfloor \mathcal{I} \subset \mathcal{I}$ implies that the characteristic solutions do not satisfy any further constraints. Elementary calculations show that our results are in complete agreement with the classical results.

In order to show the applicability of the method to nonlinear systems of first-order PDE, I consider the problem of solving the system

$$\partial_x f + \partial_t g = 1, \qquad (\partial_t f)\partial_x g = 4 \tag{6.19}$$

Note that differentiation of (6.19) and elimination of the various cross derivatives gives

$$(\partial_x g)^2 \partial_t \partial_t g = 4 \partial_x \partial_x g, \qquad (\partial_t f)^2 \partial_x \partial_x f = 4 \partial_t \partial_t f \tag{6.20}$$

Thus, (6.19) define a Bäcklund transformation that relates the two nonlinear wave equations (6.20). Accordingly, if we solve the system (6.19), then we will also have solved the two nonlinear wave equations (6.20) simultaneously.

A convenient system of local coordinates on graph space is $\{x, t, f, g\}$, and we must have the contact 1-forms

$$C^f = df - y_x^f\, dx - y_t^f\, dt, \qquad C^g = dg - y_x^g\, dx - y_t^g\, dt \tag{6.21}$$

The system of PDE (6.19) can be encoded in terms of the 0-forms

$$H^1 = y_x^f + y_t^g - 1, \qquad H^2 = y_t^f y_x^g - 4 \tag{6.22}$$

The fundamental ideal for this system is therefore

$$\mathcal{I} = I\{C^f, dC^f, C^g, dC^g, H^1, dH^1, H^2, dH^2\} \tag{6.23}$$

A straightforward calculation shows that $\mathcal{I}$ admits an isovector that is generated by

$$V_G = ax\partial_x + t\partial_t + (af + bx)\partial_f + (g - bt)\partial_g \tag{6.24}$$

for any choices of the constants (a, b); that is, $V = \mathbf{pr}^1(V_G) \in TI$ because $N > 1$. The forms of the components of V_G suggest that we consider initial data maps ψ of the form

$$\begin{gathered} \psi\colon \quad \mathbb{R} \to K \quad | \quad x = u, \quad t = 1, \quad f = A(u), \quad g = B(u) \\ y^f_x = \psi^f_x(u), \qquad y^f_t = \psi^f_t(u), \qquad y^g_x = \psi^g_x(u), \qquad y^g_t = \psi^g_t(u) \end{gathered} \tag{6.25}$$

because the transversality condition $\psi^*(V_G \rfloor \mu) \neq 0$ is then satisfied globally. An elementary calculation based on (6.25) and the definitions of the contact 1-forms shows that $\psi^* C^f = \psi^* C^g = 0$ gives

$$\psi^f_x = A'(u) = \frac{dA}{du}, \qquad \psi^f_x = B'(u) = \frac{dB}{du} \tag{6.26}$$

while $\psi^*(V \rfloor C^f) = \psi^*(V \rfloor C^g) = 0$ yields

$$\psi^f_t = a(A - uA') + bu, \qquad \psi^g_t = B - auB' - b \tag{6.27}$$

The remaining initial data constraints $\psi^* H^f = \psi^* H^g = 0$ give the reduced field equations

$$A' + B - auB' = 1 + b, \qquad \{a(A - uA') + bu\}B' = 4 \tag{6.28}$$

Note that the reduced field equations (6.28) have nontrivial solutions. Indeed,

$$A(u) = Lu - Ku\ln(u), \qquad B(u) = 1 + b + (1 + a)K - L + K\ln(u) \tag{6.29}$$

satisfy (6.28) for any choice of the integration constant L provided K is a root of the quadratic equation

$$aK^2 + bK - 4 = 0$$

The orbital equations for the characteristics of this problem are

$$\frac{dX}{d\tau} = aX, \qquad \frac{dT}{d\tau} = T, \qquad \frac{dF}{d\tau} = aF + bX, \qquad \frac{dG}{d\tau} = G - bT \tag{6.30}$$

When these are solved subject to the initial data (generated by ψ)

$$X(0) = u, \qquad T(0) = 1, \qquad F(0) = A(u), \qquad G(0) = B(u) \tag{6.31}$$

for any $A(u)$, $B(u)$ that satisfy (6.28), a characteristic solution of the original problem is given in implicit form by

$$x = ue^{a\tau}, \qquad t = e^{\tau} \tag{6.32}$$

$$f = \{A(u) + bu\tau\}e^{a\tau}, \qquad g = \{B(u) - b\tau\}e^{\tau} \tag{6.33}$$

This characteristic solution is not a general solution because $V \rfloor \mathcal{I}$ is not contained in the fundamental ideal. In particular, we know that any characteristic solution map Ψ will necessarily satisfy the constraints

$$\Psi^*(V \rfloor C^f) = \Psi^*(V \rfloor C^g) = 0$$

For the problem at hand, these differential constraints evaluate to

$$(ax\partial_x + t\partial_t)\langle f \rangle = af + bx, \qquad (ax\partial_x + t\partial_t)\langle g \rangle = g - bt \tag{6.34}$$

namely, a quasilinear first-order system with the same principal part whose characteristics are precisely the characteristics of our given nonlinear system. In fact, one may take the view that our nonlinear system has characteristic solutions precisely because we have imposed the constraints (6.34) which have characteristic solutions; that is, our nonlinear system inherits its characteristics from the imposed constraints.

Characteristic solutions to systems of second-order PDE can be constructed once an appropriate isovector of the fundamental ideal is known. Since $V \rfloor \mathcal{I}$ will not be contained in $\mathcal{I}$, in general, the characteristic solutions will satisfy a nontrivial system of differential constraints. Characteristic solutions will thus constitute only a subset of solutions to the given system of PDE.

7. HIGHER-ORDER EQUATIONS

A nonlinear PDE of third order that is frequently studied is the (homogeneous) Korteweg-de Vries equation $\partial_t\phi + \partial_x^3\phi + \phi\,\partial_x\phi = 0$. In order to show how to obtain characteristic solutions for higher-order systems, I consider the problem of solving the *inhomogeneous* KdV equation

$$\partial_t\phi + \partial_x^3\phi + \phi\,\partial_x\phi = F(x, t, \phi) \tag{7.1}$$

where restrictions on the functional form of $F(x, t, \phi)$ will be given shortly. The graph space for this problem is taken to have local coordinates $\{x, t, \phi\}$. In view of the various derivatives that occur in (7.1), we will need the contact 1-forms

$$\begin{gathered} C = d\phi - y_x\,dx - y_t\,dt, \qquad C_x = dy_x - y_{xx}\,dx - y_{xt}\,dt \\ C_{xx} = dy_{xx} - y_{xxx}\,dx - y_{xxt}\,dt \end{gathered} \tag{7.2}$$

in which case the inhomogeneous KdV equation (7.1) is encoded through the 0-form

$$H = y_t + y_{xxx} + \phi y_x - F(x, t, \phi) \tag{7.3}$$

The fundamental ideal is therefore given by

$$\mathcal{I} = I\{C, dC, C_x, dC_x, C_{xx}, dC_{xx}, H, dH\} \tag{7.4}$$

Olver (1986) has shown that the homogeneous KdV equation admits an isovector field that is generated by the third-order prolongation of

$$V_g = (a + kt + mx)\partial_x + (1 + 3mt)\partial_t + (k - 2m\phi)\partial_\phi \tag{7.5}$$

where (a, k, m) are constants. It is a lengthy, but straightforward task to determine the possible forms of $F(x, t, \phi) = \hat{F}(x, t, \phi; a, k, m)$ for which V_g also generates an isovector of the inhomogeneous KdV equation. For example, with $k = m = 0$, we must have $F(x, t, \phi) = \mathcal{F}(x - at, \phi)$. We shall assume that this has been done and that the function $F(x, t, \phi)$ that appears on the right-hand side of (7.1) and (7.3) has been chosen in accordance with this procedure. The isovector field generated by V_g is then an isovector of the fundamental ideal $\mathcal{I}$ for the inhomogeneous KdV equation.

Noting that the isovector field generated by V_g is a prolongation of a vector field on graph space, the essential information about an initial data map ψ is determined by the restriction of its range to graph space,

$$\psi \quad | \quad x = u, \quad t = 0, \quad \phi = \psi(u) \tag{7.6}$$

These serve as the initial data for the orbital equations of V_g on graph space:

$$\frac{dX}{d\tau} = a + kT + mX, \qquad X(0) = u \tag{7.7}$$

$$\frac{dT}{d\tau} = 1 + 3mT, \qquad T(0) = 0 \tag{7.8}$$

$$\frac{d\Phi}{d\tau} = k - 2m\Phi, \qquad \Phi(0) = \psi(u) \tag{7.9}$$

in which case the characteristic solution is given in implicit form by $x = X(u, \tau)$, $t = T(u, \tau)$, $\phi = \Phi(u, \tau)$. The initial data for the various y's are to be determined by satisfaction of the initial data conditions. The only place where these will be used, however, is in the evaluation of the reduced field equation. We will therefore only have to determine $\psi^* y_x$, $\psi^* y_t$, and $\psi^* y_{xxx}$. This is an obvious saving of work, and is always present whenever the isovector field is a prolongation of a vector field on graph space.

Since $\mu = dx \wedge dt$, $\psi^*(V_g \rfloor \mu) = -du$ and hence (V_g, ψ) satisfy the transversality condition for all $u \in \mathbb{R}$. The domain of the initial data map may thus be taken to be all of $\mathbb{R}$. The initial data condition $\psi^* C = 0$ gives the evaluation

$$\psi^* y_x = \psi'(u) \tag{7.10}$$

where the prime denotes differentiation with respect to u. Use of (7.6) and (7.10) then shows that the initial data condition $\psi^*(V_g \rfloor C) = 0$ gives

$$\psi^* y_t = k - 2m\psi - (a + mu)\psi' \tag{7.11}$$

It is then a simple matter to see that $\psi^*(C_{xx}) = 0$ gives

$$\psi^* y_{xxx} = \psi'''(u) \tag{7.12}$$

The reduced field equations are obtained from $\psi^* H = 0$, $\psi^*(V \rfloor H) = 0$. Since H is a 0-form, $V \rfloor H = 0$, and hence the *reduced field equation* is given by $\psi^* H = 0$; namely

$$\psi''' + (\psi - a - mu)\psi' - 2m\psi + k = F(u, 0, \psi) \tag{7.13}$$

The reader should note that this reduced field equation has been obtained without first having to integrate the orbital equations in order to compute a complete system of orbital invariant functions.

Characteristic solutions of the inhomogeneous KdV equation are given in implicit form by solving the orbital equations (7.7)-(7.9) for any initial data $\Phi(0) = \psi(u)$ that satisfy the reduced field equation (7.13). In the case $k = m = 0$, $a > 0$, we have $F(x, t, \phi) = \mathcal{F}(x - at, \phi)$ and (7.13) reduces to

$$\psi''' + (\psi - a)\psi' = \mathcal{F}(u, \psi) \tag{7.14}$$

When $\mathcal{F} = 0$, it is a simple matter to show that we obtain the "one-soliton" solution

$$\phi = 3a \operatorname{sech}^2\left(\frac{\sqrt{a}}{2}(x - at) + \delta\right) \tag{7.15}$$

because $X = u + a\tau$, $T = \tau$, $\Phi = \psi(u)$ in this case. On the other hand, even for $\mathcal{F}(u, \psi) = f(u)$, we can only obtain a first quadrature of (7.14). We therefore have to solve the nonlinear second-order differential equation

$$\psi'' + \tfrac{1}{2}\psi^2 - a\psi + K = g(u), \qquad g'(u) = f(u) \tag{7.16}$$

in order to find characteristic solutions of the inhomogeneous KdV equation when $F(x, t, \phi) = g'(x - at)$.

8. COMPARISON WITH GROUP INVARIANT (SIMILARITY) METHODS

Olver (1986, Chapter 3) gives a detailed treatment of the method of constructing group invariant (similarity) solutions to systems of PDE. An

elementary comparison of Olver's results with those presented here shows that the two methods agree whenever the isovector V (one-parameter symmetry group) of the fundamental ideal is a prolongation of a vector field on graph space (remember that this is always the case when $N>1$, but need not hold when $N=1$). The methodologies of the two methods are quite different, however, and these differences can be crucial.

First, if the isovector V is not a Cauchy characteristic vector field of the fundamental ideal (i.e., $V\rfloor\mathcal{I}$ is not contained in $\mathcal{I}$), the method of characteristics shows that any characteristic solution map Ψ necessarily satisfies the differential constraints

$$\Psi^*(V\rfloor C^\alpha)=0, \qquad 1\leq\alpha\leq N \tag{8.1}$$

while this is not manifestly evident in the group invariant method. The characteristic method thus provides an understanding of why we obtain general solutions for a single first-order PDE and for quasilinear systems with the same principal part (when $V\rfloor\mathcal{I}\subset\mathcal{I}$), and why we do not obtain general solutions for systems of nonlinear first-order and higher-order PDE. I note in passing that the group invariant method discussed by Olver cannot be used for a single nonlinear PDE in a single unknown because the isovector field for such a problem is not a prolongation of a vector field on graph space.

Second, the computational procedures of the two methods differ markedly. Olver (1986, p. 192) gives a seven-step procedure for constructing group invariant solutions. The fourth step of this procedure requires the explicit construction of a complete set of functionally independent invariants of the group. For the case of a one-parameter group generated by an isovector V, this means that the orbital equations of V must be solved explicitly. Now, it is the exception rather than the rule that a system of orbital equations can be solved explicitly, and hence it is the exception rather than the rule that a complete set of functionally independent invariants can be constructed explicitly. If such a system of independent invariants cannot be constructed, then the method given by Olver comes to a screeching halt; that is, one is unable to solve the unknown invariants for a system of primary and parametric independent variables and the calculation of the reduced system of PDE cannot be performed.

Knowledge of the isovector V and the fundamental ideal and an initial data map ψ that satisfy the transversality condition allows us to compute all of the initial data constraints

$$\psi^*C^\alpha=0, \qquad \psi^*(V\rfloor C^\alpha)=0, \qquad \psi^*C^\alpha_i=0,\ldots \tag{8.2}$$

and the reduced field equations

$$\psi^*H^a=0, \qquad \psi^*(V\rfloor H^a)=0 \tag{8.3}$$

The method of characteristics thus separates the problem of solving the

orbital equations from the problem of computing the reduced field equations, while the group invariant method does not. We obtain explicit evaluations of the reduced field equations without having to solve the orbital equations and without having to decide on a system of parametric independent variables and then eliminate them from the field equations.

Both methods ultimately require us to solve the orbital equations of the isovector V in order to obtain explicit solutions. There is a further difference, however, for we could resort to numerical integration of the orbital equations of V in the method presented here because we can obtain the reduced field equations before such integrations are performed. Numerical integration of the orbital equations in the group invariant method is not practical expressly because such numerical techniques are incapable of obtaining a complete system of (precise) invariants of the flow of V. The method of characteristics is thus capable of using approximate numerical methods where necessary.

The characteriwtic method allows us to "watch" the unfolding of the characteristic solution from its initial data, while the group invariant method does not provide this capability directly. There is, of course, a price to be payed for this capability. It takes the form of the requirement that we choose an initial data map that satisfies the transversality condition. The group invariant method likewise contains an implicit transversality condition. It is contained in the requirement that we are able to solve the complete system of invariants for primary and parametric independent variables, and that the parametric independent variables can be eliminated in the reduced field equations.

Both the characteristic method and the group invariant method lead to systems of reduced field equations that must be solved explicitly in order to obtain solutions to the original problem. If $n>2$, the reduced field equations that result from an isovector V will be partial differential equations with one less independent variable. The space of new independent variables in the group invariant method is not specifically tied down by that method, while we have the explicit domain D_{n-1} of the initial data map ψ for the new space of independent variables in the characteristic method. Since the reduced field equations are actually field equations that are pulled back to D_{n-1} by the action of ψ^*, we obtain a well-defined new problem on the domain D_{n-1}. We can then apply the method of characteristics to this new problem by finding an isovector of the fundamental ideal for this new problem. Successive reductions are thus directly available with the method of characteristics without having to solve the orbital equations at each stage. The group invariant method also provides for successive reductions, but they are of an implicit nature and require explicit determination of a complete system of independent invariants at each stage.

Finally, note that the method of characteristics is a "dimension reduction" method, while the Cartan-Kähler (Cartan, 1984) method for involutive systems is a "dimension induction" method. The two methods, when applicable, should thus be complementary.

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